Formula for Vector Rotation in Arbitrary Planes in $\Re^n$

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1 Introduction

We derive a formula for the rotation of a vector in an arbitrary plane that is applicable to $\Re^n$ for all $n \geq 2$. Since all general rotations can be decomposed into a sequence of plane rotations, this formula is applicable to rotations in general.

Although in $\Re^3$ every plane of rotation is uniquely defined by an axis, this is not true in any other dimension. Hence, a better way of defining rotation is by specifying the *plane* in which we are rotating, and a point on this plane, the *center of rotation,* about which we are rotating.

Formulæ for vector rotation are well-known. Here, however, we desire a formula applicable to $\Re^n$ for all $n \geq 2$, and therefore need a formulation purely in terms of vector operations, and not involving any explicit Cartesian coordinates.

2 Assumptions

Let $\mathbf{v} \in \Re^n$ be a vector. We shall denote the image of $\mathbf{v}$ under rotation in a plane P by an angle of θ as $\mathrm{rot}_{P,\theta}(\mathbf{v})$. Without loss of generality, we make the following assumptions:

1. The actual plane of rotation P is parallel to a plane P_0 that passes through the origin.

2. The center of rotation C lies in the $(n-2)$-hyperplane that intersects P_0 at the origin. That is, the desired rotation in P about C is isomorphic via translation to a rotation in the plane P_0 about the origin.

3. P_0 is the span of two orthogonal unit vectors $\mathbf{x}, \mathbf{y} \in \Re^n$. Since planes of rotation are always 2-dimensional, $\mathbf{x}$ and $\mathbf{y}$ are sufficient to fully specify P_0.

4. We define the sense of rotation as rotating $\mathbf{x}$ into $\mathbf{y}$. That is,

$$\mathrm{rot}_{P_0,\pi/2}(\mathbf{x}) = \mathbf{y}$$

3 Derivation

3.1 Reduction to Rotation in P_0

Let θ be the angle by which we wish to rotate $\mathbf{v}$. We shall achieve the rotation by rotating the projection of $\mathbf{v}$ in P_0, and then mapping the result back to plane in which $\mathbf{v}$ lies.

Let $\mathbf{v}_p$ be the projection of $\mathbf{v}$ onto P_0:

$$\mathbf{v}_p = (\mathbf{v} \cdot \mathbf{x})\mathbf{x} + (\mathbf{v} \cdot \mathbf{y})\mathbf{y} \tag{1}$$

Theorem 1 *$(\mathbf{v} - \mathbf{v}_p)$ is orthogonal to P_0.*

Proof. We show that $(\mathbf{v} - \mathbf{v}_p)$ is orthogonal to $\mathbf{x}$ and $\mathbf{y}$.

$$\begin{aligned}
(\mathbf{v} - \mathbf{v}_p) \cdot \mathbf{x} &= \mathbf{v} \cdot \mathbf{x} - \mathbf{v}_p \cdot \mathbf{x} \\
&= \mathbf{v} \cdot \mathbf{x} - ((\mathbf{v} \cdot \mathbf{x})\mathbf{x} + (\mathbf{v} \cdot \mathbf{y})\mathbf{y}) \cdot \mathbf{x} \\
&= \mathbf{v} \cdot \mathbf{x} - ((\mathbf{v} \cdot \mathbf{x})(\mathbf{x} \cdot \mathbf{x}) + (\mathbf{v} \cdot \mathbf{y})(\mathbf{y} \cdot \mathbf{x}))
\end{aligned}$$

But since $\mathbf{x}$ and $\mathbf{y}$ are orthogonal unit vectors,

$$\begin{aligned}
(\mathbf{v} - \mathbf{v}_p) \cdot \mathbf{x} &= \mathbf{v} \cdot \mathbf{x} - ((\mathbf{v} \cdot \mathbf{x})\|\mathbf{x}\|^2 + 0) \\
&= \mathbf{v} \cdot \mathbf{x} - \mathbf{v} \cdot \mathbf{x} \\
&= 0
\end{aligned}$$

Similarly,

$$\begin{aligned}
(\mathbf{v}-\mathbf{v}_p)\cdot\mathbf{y} &= \mathbf{v}\cdot\mathbf{y}-\mathbf{v}_p\cdot\mathbf{y} \\
&= \mathbf{v}\cdot\mathbf{y}-((\mathbf{v}\cdot\mathbf{x})\mathbf{x}+(\mathbf{v}\cdot\mathbf{y})\mathbf{y})\cdot\mathbf{y} \\
&= \mathbf{v}\cdot\mathbf{y}-((\mathbf{v}\cdot\mathbf{x})(\mathbf{x}\cdot\mathbf{y})+(\mathbf{v}\cdot\mathbf{y})(\mathbf{y}\cdot\mathbf{y})) \\
&= \mathbf{v}\cdot\mathbf{y}-(0+(\mathbf{v}\cdot\mathbf{y})\|\mathbf{y}\|^2) \\
&= \mathbf{v}\cdot\mathbf{y}-\mathbf{v}\cdot\mathbf{y} \\
&= 0 \quad \blacksquare
\end{aligned}$$

This means that $(\mathbf{v}-\mathbf{v}_p)$ is the component of $\mathbf{v}$ that is orthogonal to P_0. Hence, it is unchanged by the rotation in P. Therefore, the image of $\mathbf{v}$ under rotation by θ in P is equal to the image of $\mathbf{v}_p$ under rotation by θ in P_0 plus $(\mathbf{v}-\mathbf{v}_p)$. That is,

$$\mathrm{rot}_{P,\theta}(\mathbf{v}) = \mathrm{rot}_{P_0,\theta}(\mathbf{v}_p) + (\mathbf{v}-\mathbf{v}_p) \tag{2}$$

In other words, it is sufficient to compute the rotation of $\mathbf{v}_p$ in P_0 by θ, and adding $(\mathbf{v}-\mathbf{v}_p)$ to the result.

3.2 Rotation in P_0

From (1), we see that the coordinates of $\mathbf{v}_p$ in P_0 with respect to $\mathbf{x}$ and $\mathbf{y}$ are $\mathbf{v}\cdot\mathbf{x}$ and $\mathbf{v}\cdot\mathbf{y}$, respectively. Let ϕ be the angle between $\mathbf{v}_p$ and $\mathbf{x}$. Then:

$$\mathbf{v}\cdot\mathbf{x} = \|\mathbf{v}_p\|\cos\phi \tag{3}$$

$$\mathbf{v}\cdot\mathbf{y} = \|\mathbf{v}_p\|\sin\phi \tag{4}$$

The resulting vector after rotating $\mathbf{v}_p$ by θ makes an angle of $(\phi+\theta)$ with $\mathbf{x}$. Therefore:

$$\mathrm{rot}_{P_0,\theta}(\mathbf{v}_p) = \|\mathbf{v}_p\|\cos(\phi+\theta)\mathbf{x} + \|\mathbf{v}_p\|\sin(\phi+\theta)\mathbf{y}$$

Using the trigonometric additive identities and applying (3) and (4), we have:

$$\begin{aligned}
\|\mathbf{v}_p\|\cos(\phi+\theta) &= \|\mathbf{v}_p\|(\cos\phi\cos\theta-\sin\phi\sin\theta) \\
&= (\|\mathbf{v}_p\|\cos\phi)\cos\theta-(\|\mathbf{v}_p\|\sin\phi)\sin\theta \\
&= (\mathbf{v}\cdot\mathbf{x})\cos\theta-(\mathbf{v}\cdot\mathbf{y})\sin\theta \qquad (5)
\end{aligned}$$

And:

$$
\begin{aligned}
\|\mathbf{v}_p\| \sin(\phi + \theta) &= \|\mathbf{v}_p\|(\sin\phi\cos\theta + \cos\phi\sin\theta) \\
&= (\|\mathbf{v}_p\|\sin\phi)\cos\theta + (\|\mathbf{v}_p\|\cos\phi)\sin\theta \\
&= (\mathbf{v}\cdot\mathbf{y})\cos\theta + (\mathbf{v}\cdot\mathbf{x})\sin\theta
\end{aligned} \tag{6}
$$

Therefore, the image of $\mathbf{v}_p$ under rotation in P_0 by θ is:

$$
\begin{aligned}
\mathrm{rot}_{P_0,\theta}(\mathbf{v}_p) &= ((\mathbf{v}\cdot\mathbf{x})\cos\theta - (\mathbf{v}\cdot\mathbf{y})\sin\theta)\mathbf{x} + \\
&\quad ((\mathbf{v}\cdot\mathbf{y})\cos\theta + (\mathbf{v}\cdot\mathbf{x})\sin\theta)\mathbf{y}
\end{aligned} \tag{7}
$$

3.3 Rotation in P

Finally, substituting (1) and (7) into (2), we get:

$$
\begin{aligned}
\mathrm{rot}_{P,\theta}(\mathbf{v}) &= ((\mathbf{v}\cdot\mathbf{x})\cos\theta - (\mathbf{v}\cdot\mathbf{y})\sin\theta)\mathbf{x} + \\
&\quad ((\mathbf{v}\cdot\mathbf{y})\cos\theta + (\mathbf{v}\cdot\mathbf{x})\sin\theta)\mathbf{y} + \\
&\quad \mathbf{v} - ((\mathbf{v}\cdot\mathbf{x})\mathbf{x} + (\mathbf{v}\cdot\mathbf{y})\mathbf{y}) \\
&= \mathbf{v} + [(\mathbf{v}\cdot\mathbf{x})\cos\theta - (\mathbf{v}\cdot\mathbf{y})\sin\theta - (\mathbf{v}\cdot\mathbf{x})]\,\mathbf{x} + \\
&\quad [(\mathbf{v}\cdot\mathbf{y})\cos\theta + (\mathbf{v}\cdot\mathbf{x})\sin\theta - (\mathbf{v}\cdot\mathbf{y})]\,\mathbf{y} \\
&= \mathbf{v} + [(\mathbf{v}\cdot\mathbf{x})(\cos\theta - 1) - (\mathbf{v}\cdot\mathbf{y})\sin\theta]\,\mathbf{x} + \\
&\quad [(\mathbf{v}\cdot\mathbf{y})(\cos\theta - 1) + (\mathbf{v}\cdot\mathbf{x})\sin\theta]\,\mathbf{y}
\end{aligned} \tag{8}
$$

Equation (8) gives us a formula for rotating the vector $\mathbf{v}$ by θ in the plane defined by $\mathbf{x}$ and $\mathbf{y}$. Since it is written entirely in terms of vector operations involving arbitrary-dimensional vectors $\mathbf{v}$, $\mathbf{x}$, and $\mathbf{y}$, it can be applied to all $\Re^n$ for $n \geq 2$.

It can also be written in the following (notationally abusive) matrix form, which is visually more appealing:

$$
\mathrm{rot}_{P,\theta}(\mathbf{v}) = \mathbf{v} + \begin{bmatrix} \mathbf{x} & \mathbf{y} \end{bmatrix} \begin{bmatrix} (\cos\theta - 1) & -\sin\theta \\ \sin\theta & (\cos\theta - 1) \end{bmatrix} \begin{bmatrix} \mathbf{v}\cdot\mathbf{x} \\ \mathbf{v}\cdot\mathbf{y} \end{bmatrix} \tag{9}
$$

In this form, its analogy with the familiar 2-dimensional rotation matrix is clear. The extra -1 terms are because $\mathbf{v}$ appears as a separate term in the equation. When $\mathbf{v}$ lies on the plane P_0, the equation reduces to the familiar 2-dimensional form.